Prof. Dr. Daniel Huybrechts
Dr. Gebhard Martin Winter term 2020/21

Exercises, Algebraic Geometry I – Week 10

Exercise 60. *Invertible sheaves* (3 points)
Let $\varphi\colon \mathcal{L} \to \mathcal{M}$ be a homomorphism of invertible sheaves on a scheme X.

(i) Show that φ is an isomorphism if φ is surjective.

(ii) Give an example where φ is injective but not an isomorphism.

Exercise 61. *Trivializing invertible sheaves* (3 points)
Assume $\mathcal{L}$ is an invertible sheaf on an integral scheme X which is projective over an algebraically closed field k. Show that L is trivial if and only if $H^0(X, \mathcal{L}) \neq 0 \neq H^0(X, \mathcal{L}^*)$.

Exercise 62. *Tensor products of ample line bundles* (3 points)
Let $f : X \to Y$ and $g : Y \to Z$ be two morphisms of schemes. Let $\mathcal{L}$ and $\mathcal{M}$ be f-relatively very ample invertible sheaves on X, and let $\mathcal{N}$ be a g-relatively very ample invertible sheaf on Y.

(i) Show that $\mathcal{L} \otimes \mathcal{M}$ is f-relatively very ample.

(ii) Show that $\mathcal{L} \otimes f^*\mathcal{N}$ is $(g \circ f)$-relatively very ample.

Exercise 63. *Ramified coverings* (5 points)
Consider a homogeneous polynomial $f \in k[x_0, \ldots, x_n]$ of degree d and the two closed subschemes $X = V_+(f) \subset \mathbb{P}^n_k$ and $Y = V_+(f - x^d_{n+1}) \subset \mathbb{P}^{n+1}_k$. Here, k is an algebraically closed field of characteristic $\mathrm{char}(k) = p \nmid d$. Show that there exists a morphism $g\colon Y \to \mathbb{P}^n_k$ with the following properties:

(i) Restricted to the intersection $Y \cap V_+(x_{n+1}) = V_+(f - x^d_{n+1}, x_{n+1}) \subset \mathbb{P}^{n+1}_k$ the morphism g yields an isomorphism with X.

(ii) For every closed point in the complement of $X \subset \mathbb{P}^n_k$ the fibre of g is a reduced scheme consisting of exactly d k-rational points.

(iii) There exists a divisor D on Y such that the invertible sheaf $\mathcal{O}(D)$ associated to it is a d-th root of the pull-back of the invertible sheaf associated to $X \subset \mathbb{P}^n_k$, i.e. $\mathcal{O}(dD) \cong g^*\mathcal{O}(X)$.

Due Friday 22 January 2021.

Exercise 64. *Example linear systems* (4 points)
Consider $s_0 := x_0^2, s_1 := x_1^2, s_2 := x_0x_1, s_3 := x_0x_2, s_4 := x_1x_2, s_5 := x_2^2 \in H^0(\mathbb{P}_k^2, \mathcal{O}(2))$, where k is an algebraically closed field.

(i) Determine the maximal open set $U \subset \mathbb{P}_k^2$ on which the map to $\mathbb{P}_k^4$ determined by s_0, s_1, s_2, s_3, s_4 is regular.

(ii) Use the local criterion to check whether the map to $\mathbb{P}_k^4$ determined by s_0, s_1, s_2, s_3, s_5 is a closed immersion if $\operatorname{char}(k) \neq 2$. What happens if $\operatorname{char}(k) = 2$?

The last exercise is not strictly necessary for the understanding of the lectures at this point.

Exercise 65. *Locally free sheaves on* $\mathbb{P}^1$ (6 extra points)
Show that every locally free coherent sheaf $\mathcal{F}$ of rank $r \geq 1$ on $\mathbb{P}_k^1$, where k is any field, is of the form

$$\mathcal{F} \cong \mathcal{O}(a_1) \oplus \cdots \oplus \mathcal{O}(a_r),$$

where the a_i are uniquely determined by $\mathcal{F}$ up to permutation, by following the steps below:

(i) Show that the set of $n \in \mathbb{Z}$ such that there is an injection $i : \mathcal{O}(n) \hookrightarrow \mathcal{F}$ is non-empty and bounded from above. Show that if n is maximal, then $\mathcal{F}/\mathcal{O}(n)$ is locally free of rank $r - 1$.

(ii) Show that if $\mathcal{F}$ has rank 2 and the maximal n in (i) is -1, then $\mathcal{F}/\mathcal{O}(-1) \cong \mathcal{O}(m)$ for some $m < 0$. Deduce that, for arbitrary locally free coherent $\mathcal{F}$ of rank r, there exists a filtration

$$0 = \mathcal{F}_0 \subset \mathcal{F}_1 \subset \mathcal{F}_2 \subset \ldots \subset \mathcal{F}_r = \mathcal{F}$$

such that $\mathcal{F}_i$ is locally free of rank i and such that $\mathcal{F}_i/\mathcal{F}_{i-1} \cong \mathcal{O}(a_i)$ with $a_1 \geq \cdots \geq a_r$.

(iii) Prove that for every $\mathcal{F}$ with a filtration as in (ii) we have $\mathcal{F} \cong \mathcal{O}(a_1) \oplus \ldots \oplus \mathcal{O}(a_r)$ and the a_i are uniquely determined by $\mathcal{F}$.

(iv) Deduce the following normal form for matrices over $k[t, t^{-1}]$:

Let M be an $r \times r$ matrix over $k[t, t^{-1}]$ with determinant t^n for some $n \in \mathbb{Z}$. Then, there exist matrices $A \in \mathrm{GL}_r(k[t^{-1}])$ and $B \in \mathrm{GL}_r(k[t])$ such that

$$A \cdot M \cdot B = \begin{pmatrix} t^{a_1} & & & 0 \\ & t^{a_2} & & \\ & & \ddots & \\ 0 & & & t^{a_r} \end{pmatrix}$$

with $a_1 \geq a_2 \geq \ldots \geq a_r$, $a_i \in \mathbb{Z}$, and the a_i are uniquely determined by M.

(Hint: First, note that every locally free coherent sheaf on $\mathbb{A}_k^1$ is free and use M to glue $k[t]^r$ with $k[t^{-1}]^r$ over $k[t, t^{-1}]$ to a locally free sheaf on $\mathbb{P}_k^1$.)